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Necessary and Sufficient Conditions for Transversals of Countable Set Systems

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This paper proves a conjecture of C. St. J. A. Nash-Williams giving necessary and sufficient conditions for an arbitrary countable system of sets to have a transversal.

1. INTRODUCTION

Let $\mathbf{B} = \langle B_i \mid i \in I \rangle$ be a system of subsets of X indexed by the set I. We write $|\mathbf{B}| = |I|$ to denote the cardinality of the system. A *transversal* of $\mathbf{B}$ is a 1–1 function $\Phi : I \to X$ such that $\Phi(i) \in B_i (i \in I)$. The element $\Phi(i)$ is called the *representative* for the set B_i in the transversal Φ and

$$\langle \Phi(i) \mid i \in I \rangle$$

is a *system of distinct representatives* of $\mathbf{B}$. For $Y \subseteq X$, let

$$\mathbf{B}(Y) = \langle B_i \mid i \in I \;\&\; B_i \subseteq Y \rangle$$

denote the subsystem of $\mathbf{B}$ containing all those members of $\mathbf{B}$ which are subsets of Y. In any transversal of $\mathbf{B}$, the representatives for the members of $\mathbf{B}(Y)$ are distinct elements of Y and so an obvious *necessary* condition for the existence of a transversal of $\mathbf{B}$ is that

$$|Y| \geq |\mathbf{B}(Y)| \quad (\forall Y \subseteq X). \tag{1.1}$$

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A well-known theorem of P. Hall [5] (proved in an equivalent formulation by D. König [6]) asserts that (1.1) is also *sufficient* for the existence of a transversal of **B** if

$$|\mathbf{B}| < \aleph_0 . \tag{1.2}$$

M. Hall [4] extended this result by showing that (1.1) is sufficient also in the case when $|\mathbf{B}|$ is arbitrary and

$$|B_i| < \aleph_0 \ (\forall i \in I). \tag{1.3}$$

One of the finiteness conditions (1.2) or (1.3) is essential for this result. For example, the system $\mathbf{B} = \langle B_n \mid n < \omega \rangle$, where

$$B_0 = \{0, 1, 2, \ldots\}, \qquad B_n = \{n - 1\} (1 \leqslant n < \omega),$$

satisfies (1.1) but does not have a transversal. R.A. Brualdi and E.B. Scrimger [2] and J. Folkman [3] and D.R. Woodall [9] independently gave stronger conditions than (1.1) which are both necessary and sufficient for the existence of a transversal of **B** in the case where

$$|\{i \in I \mid |B_i| \geqslant \aleph_0\}| < \aleph_0 \tag{1.4}$$

is satisfied. But even for this special case the conditions of [2, 3 and 9] are fairly complicated and the general problem remains unsolved.

Recently, C. St. J.A. Nash-Williams [8] stated conditions which he conjectured to be necessary and sufficient for the existence of a transversal of a system **B** which satisfies

$$|\mathbf{B}| \leqslant \aleph_0 , \tag{1.5}$$

and in this paper we verify his conjecture.

We give a brief indication of the intuitive ideas behind Nash-Williams' conjecture described more fully in [8]. Let **B** be any system, not necessarily countable, and suppose **B** does have a transversal. Then (1.1) holds and the difference

$$m_0(Y) = |Y| - |\mathbf{B}(Y)| , \tag{1.6}$$

if meaningful, measures the number of elements in Y which are left over after choosing representatives for those members of **B** which are subsets of Y. These "spare" elements may be used to represent members of **B** not in $\mathbf{B}(Y)$. Of course, the difference $|Y| - |\mathbf{B}(Y)|$ is meaningless in the case when $|Y| = |\mathbf{B}(Y)| \geqslant \aleph_0$. Following Nash-Williams, define $m_0(Y)$ by

(1.6) in the case when $|Y| < \aleph_0$ and put $m_0(Y) = \infty$ if $|Y| \geqslant \aleph_0$. Then Hall's condition (1.1) is equivalent to the condition

$$m_0(Y) \geqslant 0 \quad (\forall Y \subseteq X).$$

We are interested in obtaining an estimate for the number of elements of Y which are left over after choosing representatives for the members of $\mathbf{B}(Y)$, the "margin" of Y as it is called in [8]. Nash-Williams considered $m_0(Y)$ to be a first approximation to this "margin", and he suggested replacing m_0 by a sequence of successively more and more demanding margin functions m_α for ordinals α. The idea is illustrated by describing the step from m_0 to m_1.

Suppose f is a function defined on the subsets of X with values in $\{0, \pm 1, \pm 2, \ldots, \pm\infty\}$. If $Y \subseteq X$, we define $\mathscr{A}(Y, f)$ to be the set of all sequences $\mathbf{T} = \langle T_n \mid n < \omega \rangle$ such that

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots \subseteq Y = \bigcup_{n<\omega} T_n, \tag{1.7}$$

$$f(T_n) = f(T_0) < \infty \quad (n = 0, 1, 2, \ldots).$$

Let $Y \subseteq X$, $\mathbf{T} = \langle T_n \mid n < \omega \rangle \in \mathscr{A}(Y, m_0)$. Then $s = s_0(\mathbf{T}) = m_0(T_0) < \infty$. Let $d(\mathbf{T})$ denote the number of indices $i \in I$ such that

$$B_i \subseteq Y, \quad B_i \nsubseteq T_n \quad (\forall n < \omega). \tag{1.8}$$

Each T_n is finite by the definition of m_0, and, if Φ is any transversal of $\mathbf{B}$, the number of "spare" elements from T_n is at most s. The representatives in Φ for the members of $\mathbf{B}$ satisfying (1.8) must also be distinct elements of Y and, in view of (1.7), we should have $d(\mathbf{T}) \leqslant s_0(\mathbf{T})$, and then the number of "spare" elements from Y is at most $s_0(\mathbf{T}) - S_0(\mathbf{T})\, d(\mathbf{T})$. Accordingly, Nash-Williams defines the function

$$m_1(Y) = \begin{cases} \infty & \text{if } \mathscr{A}(Y, m_0) = \varnothing, \\ \inf\limits_{\mathbf{T} \in \mathscr{A}(Y, m_0)} (s_0(\mathbf{T}) - d(\mathbf{T})) & \text{if } \mathscr{A}(Y, m_0) \neq \varnothing. \end{cases}$$

It is almost obvious that a necessary condition for the existence of a transversal for $\mathbf{B}$ is that

$$m_1(Y) \geqslant 0 \quad (\forall Y \subseteq X). \tag{1.9}$$

This is stronger than (1.1) since $m_0(Y) \geqslant m_1(Y)$. More generally, $m_{\alpha+1}$ is obtained from m_α in a similar way to that in which m_1 is obtained from m_0 and, for a limit ordinal α, m_α is defined by putting

$$m_\alpha(Y) = \inf_{\beta<\alpha} m_\beta(Y).$$

This defines the sequence of functions $m_\alpha (\alpha \in \mathbf{On})$ by transfinite induction so that

$$m_0(Y) \geqslant m_1(Y) \geqslant \cdots \geqslant m_\alpha(Y) \geqslant \cdots (\forall Y \subseteq X).$$

It is not difficult to show (Theorem 1) that the condition

$$m_\alpha(Y) \geqslant 0 \qquad (\forall Y \subseteq X; \forall \alpha \in \mathbf{On}) \tag{1.10}$$

is necessary for $\mathbf{B}$ to have a transversal. In fact one can show (Lemma 3.1) that if Ω is the first uncountable ordinal, then $m_\alpha(Y) = m_\Omega(Y)$ for $\alpha \geqslant \Omega$ and so (1.10) is equivalent to the single condition

$$m_\Omega(Y) \geqslant 0 \qquad (\forall Y \subseteq X). \tag{1.11}$$

Our main result (Theorem 2) is that, as conjectured by Nash-Williams, (1.11) is also sufficient for the existence of a transversal provided (1.5) holds.

In Section 7 we give two examples. The first shows that (1.11) is not sufficient for the existence of a transversal of an uncountable system $\mathbf{B}$ even though the members of $\mathbf{B}$ are countable, i.e.,

$$| B_i | \leqslant \aleph_0 \qquad (\forall i \in I). \tag{1.12}$$

The second gives, for each $\alpha < \Omega$, an example of a system $\mathbf{B}$ for which (1.5) holds and

$$m_\alpha(Y) \geqslant 0 \ (\forall Y \subseteq X), \qquad m_{\alpha+1}(X) < 0.$$

This shows that Ω cannot be replaced by any smaller ordinal in condition (1.11).

In Section 8 we show that the weaker condition (1.9) is both necessary and sufficient for the existence of a transversal of $\mathbf{B}$ provided (1.4) and (1.12) hold. (This easily implies another conjecture of Nash-Williams [8] that (1.9) is necessary and sufficient for the existence of a transversal of $\mathbf{B}$ if (1.4) and (1.5) hold). The condition (1.9) is not equivalent to either of the conditions of [2], [3] or [9] since the conditions given in those papers apply to any $\mathbf{B}$ satisfying (1.4), whereas the family $\mathbf{B} = \langle B_\alpha \mid \alpha \leqslant \Omega \rangle$, with

$$B_\alpha = \{\alpha\} (\alpha < \Omega), \qquad B_\Omega = \{\beta \mid \beta < \Omega\},$$

having a single uncountable infinite member satisfies (1.9) but does not have a transversal.

2. Notation

Greek letters denote ordinal numbers; in particular, ω is the first infinite ordinal and Ω is the first uncountable ordinal. **On** denotes the class of all ordinals. The ordinal α is the set $\{\beta \mid \beta < \alpha\}$ of all smaller ordinal numbers. Capital letters denote sets and bold type capitals denote systems of sets. Properly speaking, a system of sets $\mathbf{B}$ indexed by I is a function, a set of ordered pairs, $\{\langle i, B_i\rangle \mid i \in I\}$, but we shall write simply $\mathbf{B} = \langle B_i \mid i \in I\rangle$. Some of the familiar terminology of set theory is abused by applying it to systems of sets. For example, we say B is a member of $\mathbf{B} = \langle B_i \mid i \in I\rangle$ and write $B \in \mathbf{B}$, if $B = B_i$ for some $i \in I$. The cardinality of $\mathbf{B}$ is $|\mathbf{B}| = |I|$. Let $\mathbf{B} = \langle B_i \mid i \in I\rangle$, $\mathbf{C} = \langle C_j \mid j \in J\rangle$ be two set systems. We say that $\mathbf{C}$ is a *subsystem* of $\mathbf{B}$ and write $\mathbf{C} \subseteq \mathbf{B}$ if and only if there is an injection $f\colon J \to I$ such that $C_j = B_{f(j)}(j \in J)$; in this case we write $\mathbf{B} - \mathbf{C}$ to denote the subsystem $\langle B_i \mid i \in I - f(J)\rangle$. The two systems $\mathbf{B}$ and $\mathbf{C}$ are *equal*, $\mathbf{B} = \mathbf{C}$, if there is a surjection $f\colon J \to I$ such that $C_j = B_{f(j)}$; equivalently,

$$\mathbf{B} = \mathbf{C} \Leftrightarrow \mathbf{B} \subseteq \mathbf{C} \;\&\; \mathbf{C} \subseteq \mathbf{B}.$$

In view of this we can unambiguously define the *sum* of two systems $\mathbf{B} = \langle B_i \mid i \in I\rangle$ and $\mathbf{C} = \langle C_j \mid j \in J\rangle$ to be

$$\mathbf{B} + \mathbf{C} = \langle D_k \mid k \in I \cup J'\rangle,$$

where I, J' are disjoint sets, $D_k = B_k(k \in I)$, $D_k = C_{g(k)}(k \in J')$ and g is a surjection from J' onto J. If P is a set and $\mathbf{B} = \langle B_i \mid i \in I\rangle$ is a system, we define

$$\mathbf{B}\backslash P = \langle B_i - P \mid i \in I\rangle, \qquad \mathbf{B} \cup P = \langle B_i \cup P \mid i \in I\rangle.$$

If a is an infinite cardinal number, we sometimes write $a = \infty$. Also, we define $n - \infty = -\infty$ if n is an integer; as usual, $-\infty - \infty = -\infty$. The symbol $\infty - \infty$ will not occur. The letters s, p, q, k denote numbers in the set $\bar{Z} = \{0, \pm 1, \pm 2, \ldots, \pm\infty\}$; other small latin letters (unless specifically stated otherwise) denote nonnegative integers. Thus, $p < \infty$ means that $p \in \bar{Z} - \{\infty\}$, and $n < \omega$ means $n \in \{0, 1, 2, \ldots\}$.

A *transversal* of the system $\mathbf{B} = \langle B_i \mid i \in I\rangle$ of subsets of X is a 1–1 map $\Phi\colon I \to X$ such that $\Phi(i) \in B_i(\forall i \in I)$. We sometimes call $\Phi(i)$ the *representative* of the member B_i of the system. Let Trans($\mathbf{B}$) denote the set of all transversals of $\mathbf{B}$. For $\Phi \in$ Trans($\mathbf{B}$) and $J \subseteq I$ we write $\Phi(J) = \{\Phi(j) \mid j \in J\}$. The number of elements of X not used to represent a member of $\mathbf{B}$ in the transversal Φ is denoted by

$$l(\Phi) = l(\Phi, \mathbf{B}, X) = \begin{cases} |X - \Phi(I)| & \text{if } X - \Phi(I) \text{ is finite,} \\ \infty & \text{if } X - \Phi(I) \text{ is infinite.} \end{cases}$$

If $J \subseteq I$, then as usual $\Phi \upharpoonright J$ denotes the restriction of Φ to J. We also write $\mathbf{B} \upharpoonright J$ to denote the subsystem $\langle B_i \mid i \in J \rangle$ of $\mathbf{B}$.

A *tower under* the set Y is a system $\mathbf{T} = \langle T_n \mid n < \omega \rangle$ such that

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots \subseteq Y = \bigcup_{n<\omega} T_n .$$

Let f be a function defined on the subsets of a set X with values in $\bar{Z}$. Then, for $Y \subseteq X$ let $\mathscr{A}(Y, f)$ denote the set of all towers $\mathbf{T} = \langle T_n | n < \omega \rangle$ under Y for which

$$f(T_n) = f(T_0) < \infty \qquad (\forall n < \omega).$$

Let $\mathbf{B} = \langle B_i \mid i \in I \rangle$ be a system of subsets of X. For $Y \subseteq X$ and $J \subseteq I$ we put $J(Y) = \{i \in J \mid B_i \subseteq Y\}$ and $\mathbf{B}(Y) = \mathbf{B} \upharpoonright I(Y)$. If $\mathbf{T} = \langle T_n \mid n < \omega \rangle$ is a tower under $Y \subseteq X$, then we define

$$\begin{aligned} F(\mathbf{T}, \mathbf{B}) &= \langle i \in I \,|(\exists n < \omega)(B_i \subseteq T_n)\rangle \\ D(\mathbf{T}, \mathbf{B}) &= I(Y) - F(\mathbf{T}, \mathbf{B}), \\ d(\mathbf{T}, \mathbf{B}) &= |\, D(\mathbf{T}, \mathbf{B})| \,. \end{aligned}$$

In most situations there will only be one system $\mathbf{B}$ under consideration and then we shall usually omit reference to $\mathbf{B}$ and simply write $F(\mathbf{T})$ instead of $F(\mathbf{T}, \mathbf{B})$ etc.

We now define the margin functions of Nash-Williams, m_α for $\alpha \in \mathbf{On}$, by transfinite induction on α. As before, let $\mathbf{B} = \langle B_i \mid i \in I \rangle$ be a given system of subsets of some set X. We define $m_\alpha(Y, \mathbf{B}) = m_\alpha(Y) \in \bar{Z}$ for $Y \subseteq X$ as follows.

Case 1. $\alpha = 0$. Put

$$m_\alpha(Y) = \begin{cases} |\, Y| - |\, \mathbf{B}(Y)| & \text{if } \; |\, Y| < \infty, \\ \infty & \text{if } \; |\, Y| = \infty. \end{cases}$$

Case 2. *α is a limit ordinal. Put*

$$m_\alpha(Y) = \inf_{\beta<\alpha} m_\beta(Y).$$

Case 3. $\alpha = \beta + 1$. Put

$$m_\alpha(Y) = \begin{cases} \inf\limits_{\mathbf{T} \in \mathscr{A}_\beta(Y)} (s_\beta(\mathbf{T}) - d(\mathbf{T})) & \text{if } \; \mathscr{A}_\beta(Y) \neq \varnothing, \\ \infty & \text{if } \; \mathscr{A}_\beta(Y) = \varnothing, \end{cases}$$

where $\mathscr{A}_\beta(Y) = \mathscr{A}(Y, m_\beta)$ and

$$s_\beta(\mathbf{T}) = m_\beta(T_n) \qquad (n < \omega)$$

for the tower $\mathbf{T} = \langle T_n \mid n < \omega \rangle \in \mathscr{A}_\beta(Y)$. This clearly defines $m_\alpha(Y) \in \bar{Z}$ for $\alpha \in \mathbf{On}$ and $Y \subseteq X$.

The following facts (2.1)–(2.8) about these margin functions are immediate consequences of the definition.

$$m_{\beta+1}(Y) < p < \infty \Rightarrow (\exists \mathbf{T} \in \mathscr{A}_\beta(Y))(s_\beta(\mathbf{T}) - d(\mathbf{T}) < p). \tag{2.1}$$

$$-\infty < m_{\beta+1}(Y) < \infty \Rightarrow (\exists \mathbf{T} \in \mathscr{A}_\beta(Y))(m_{\beta+1}(Y) = s_\beta(\mathbf{T}) - d(\mathbf{T})). \tag{2.2}$$

$$\beta \leqslant \alpha \Rightarrow m_\alpha(Y) \leqslant m_\beta(Y). \tag{2.3}$$

If $\bar{Z}$ is regarded as a topological space in which $+\infty$ and $-\infty$ are the only limit points, then, for a fixed set $Y \subseteq X$, $m_\alpha(Y)$ is a continuous function of α which maps $\mathbf{On}$ into $\bar{Z}$ (because $m_\alpha(Y)$ is monotonic by (2.3)). (2.4)

If α is a limit ordinal and $m_\alpha(Y) > -\infty$, then there is $\beta < \alpha$ such that $m_\beta(Y) = m_\alpha(Y)$ (because $\bar{Z}$ is discrete except at $\pm\infty$). (2.5)

For $Y \subseteq X$ there is $\alpha = \alpha(Y) < \Omega$ such that $m_\alpha(Y) = m_\Omega(Y)$ (because a monotonic Ω-sequence in $\bar{Z}$ is ultimately constant). (2.6)

If $m_{\beta+1}(Y) = m_\beta(Y)(\forall Y \subseteq X)$, then

$$m_\gamma(Y) = m_\beta(Y)(\forall Y \subseteq X \,\&\, \forall \gamma \geqslant \beta). \tag{2.7}$$

$m_\alpha(Y, \mathbf{B}) = m_\alpha(Y, \mathbf{B}(Y))$ $(\forall Y \subseteq X)$ (that is, in calculating the margin $m_\alpha(Y)$, the only members of $\mathbf{B}$ which are relevant are those which are subsets of Y). (2.8)

We write $\mathbf{B} \in \mathscr{P}(\alpha)$ if and only if

$$m_\alpha(Y) \geqslant 0 \qquad (\forall Y \subseteq X).$$

By (2.3),

$$\mathbf{B} \in \mathscr{P}(\alpha) \,\&\, \beta < \alpha \Rightarrow \mathbf{B} \in \mathscr{P}(\beta). \tag{2.9}$$

3. Lemmas for the Margin Functions

In this and the next two sections $\mathbf{B} = \langle B_i \mid i \in I \rangle$ denotes an arbitrary (fixed) family of subsets of a given set X.

Lemma 3.1. *If $\beta \geqslant \Omega$ and $Y \subseteq X$, then $m_\beta(Y) = m_\Omega(Y)$.*

Proof. In view of (2.3) and (2.7) it will be enough to prove that

$$m_{\Omega+1}(Y) \geqslant m_{\Omega}(Y)\ (\forall Y \subseteq X). \tag{3.1}$$

If $m_{\Omega+1}(Y) = \infty$, then (3.1) holds trivially. Therefore, we may assume that $m_{\Omega+1}(Y) < \infty$. Suppose $m_{\Omega+1}(Y) < p < \infty$. By (2.1) there is a tower $\mathbf{T}(p) = \mathbf{T} = \langle T_n \rangle \in \mathscr{A}_{\Omega}(Y)$ such that

$$s_{\Omega}(\mathbf{T}) - d(\mathbf{T}) < p.$$

By (2.6), for each $n < \omega$ there is $\alpha_n < \Omega$ such that $m_{\alpha_n}(T_n) = m_{\Omega}(T_n) = s_{\Omega}(\mathbf{T})$. Let $\alpha(p) = \sup_{n<\omega} \alpha_n$. Then, by (2.3),

$$m_{\gamma}(T_n) = s_{\Omega}(\mathbf{T}) \text{ for } \alpha(p) \leqslant \gamma \leqslant \Omega.$$

Now let $\alpha^* = \sup_p \alpha(p)$. Then $\alpha^* < \Omega$ and

$$m_{\alpha^*+1}(Y) \leqslant s_{\Omega}(\mathbf{T}(p)) - d(\mathbf{T}(p)) < p.$$

This holds for every p such that $m_{\Omega+1}(Y) < p < \infty$. Therefore

$$m_{\alpha^*+1}(Y) \leqslant m_{\Omega+1}(Y).$$

Now (3.1) follows since, by (2.3),

$$m_{\Omega}(Y) \leqslant m_{\alpha^*+1}(Y) \leqslant m_{\Omega+1}(Y).$$

This completes the proof of the lemma.

The following lemmas are all proved by induction on α.

LEMMA 3.2. *If $\alpha \in \mathbf{On}$ and $m_\alpha(X) < \infty$, then $|X| \leqslant \aleph_0$.*

Proof. *Case* 1. $\alpha = 0$. Then $m_0(X) < \infty$ implies that X is finite.

Case 2. α is a limit ordinal. Then there is $\beta < \alpha$ such that $m_\beta(X) < \infty$ and hence $|X| \leqslant \aleph_0$.

Case 3. $\alpha = \beta + 1$. By (2.1), if $m_\alpha(X) < \infty$, there is a tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(X)$ such that $m_\beta(T_n) < \infty$. By the induction hypothesis each T_n is denumerable and hence so is $X = \cup T_n$.

LEMMA 3.3. *Let $\mathbf{B}$ be a system of sets that satisfies*

$$m_\alpha(Y, \mathbf{B}) \geqslant -h \qquad (\forall Y \subseteq X)$$

for some fixed $h < \omega$. Let $\mathbf{B}' \subseteq \mathbf{B}$ and $Y \subseteq X$. Then

$$m_\alpha(Y, \mathbf{B}') \geqslant m_\alpha(Y, \mathbf{B}). \tag{3.2}$$

Proof. Case 1. $\alpha = 0$. The inequality follows from the fact that $|\mathbf{B}'(Y)| \leqslant |\mathbf{B}(Y)|$.

Case 2. α is a limit ordinal. Then by the induction hypothesis,

$$m_\alpha(Y, \mathbf{B}') = \inf_{\beta<\alpha} m_\beta(Y, \mathbf{B}') \geqslant \inf_{\beta<\alpha} m_\beta(Y, \mathbf{B}) = m_\alpha(Y, \mathbf{B}).$$

Case 3. $\alpha = \beta + 1$. If $m_\alpha(Y, \mathbf{B}') = \infty$, there is nothing to prove. Suppose $m_\alpha(Y, \mathbf{B}') < \infty$. Let $m_\alpha(Y, \mathbf{B}') < p < \infty$. Then there is a tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(Y, \mathbf{B}')$ such that $m_\beta(T_n, \mathbf{B}') = s < \infty$ and

$$s - d(\mathbf{T}, \mathbf{B}') < p.$$

By the induction hypothesis and (2.3) it follows that

$$-h \leqslant m_\beta(T_n, \mathbf{B}) \leqslant m_\beta(T_n, \mathbf{B}') = s.$$

Hence there is an infinite set of integers J such that

$$m_\beta(T_n, \mathbf{B}) = s' \leqslant s (\forall n \in J)$$

and the subtower $\mathbf{T}' = \langle T_j \mid j \in J \rangle \in \mathscr{A}_\beta(Y, \mathbf{B})$. Since $D(\mathbf{T}, \mathbf{B}') \subseteq D(\mathbf{T}', \mathbf{B})$, it follows that

$$m_\alpha(Y, \mathbf{B}) \leqslant s' - d(\mathbf{T}', \mathbf{B}) \leqslant s - d(\mathbf{T}, \mathbf{B}') < p.$$

This is true for any $p > m_\alpha(Y, \mathbf{B}')$ and hence the inequality (3.2) holds.

LEMMA 3.4. *If P is a finite set and $P \subseteq Y \subseteq X$, then*

$$m_\alpha(Y - P, \mathbf{B} \backslash P) = m_\alpha(Y, \mathbf{B}) - |P|. \tag{3.3}$$

Proof. Case 1. $\alpha = 0$. The equation holds since

$|Y - P| = |Y| - |P|$ and $|\mathbf{B}(Y)| = |\mathbf{B}'(Y - P)|$, where $\mathbf{B}' = \mathbf{B} \backslash P$.

Case 2. α is a limit ordinal. By the induction hypothesis

$$m_\beta(Y - P, \mathbf{B} \backslash P) = m_\beta(Y) - |P|$$

for every $\beta < \alpha$. Both sides of this equation are continuous functions of β (by 2.4), and (3.3) follows by taking the limit as $\beta \to \alpha$.

Case 3. $\alpha = \beta + 1$. Suppose that $m_\alpha(Y, \mathbf{B}) < k < \infty$. Then there exists a tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(Y)$ such that $m_\beta(T_n) = s$ ($\forall n < \omega$) and

$$s - d(\mathbf{T}, \mathbf{B}) < k.$$

Since P is a finite subset of Y, there is $m < \omega$ such that $P \subseteq T_n$ for all $n \geqslant m$. Let $\mathbf{U}$ be the tower $\langle T_n - P \mid m \leqslant n < \omega \rangle$. By the induction hypothesis,

$$m_\beta(T_n - P, \mathbf{B} \backslash P) = m_\beta(T_n, \mathbf{B}) - |P| = s - |P| \; (\forall n \geqslant m),$$

and so

$$\begin{aligned} m_\alpha(Y - P, \mathbf{B}\backslash P) &\leqslant s(\mathbf{U}, \mathbf{B}\backslash P) - d(\mathbf{U}, \mathbf{B}\backslash P) \\ &= s - |P| - d(\mathbf{T}, \mathbf{B}) < k - |P|. \end{aligned}$$

This inequality holds whenever $m_\alpha(Y, \mathbf{B}) < k < \infty$. Therefore

$$m_\alpha(Y - P, \mathbf{B}\backslash P) \leqslant m_\alpha(Y, \mathbf{B}) - |P|.$$

The reverse inequality is proved by a similar argument.

LEMMA 3.5. *Let* $\mathbf{E}$ *be a finite system of subsets of* X. *Then*

$$m_\alpha(Y, \mathbf{B} + \mathbf{E}) = m_\alpha(Y) - \mathbf{E}(Y)$$

holds for $\alpha \in \mathbf{On}$ *and* $Y \subseteq X$.

Proof. This follows by induction on α and the observation that, if $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(Y, \mathbf{B})$, then $\mathbf{E}(T_n)$ is eventually constant and

$$d(\mathbf{T}, \mathbf{B} + \mathbf{E}) = d(\mathbf{T}, \mathbf{B}) + d(\mathbf{T}, \mathbf{E}).$$

4. NECESSITY OF POSITIVE MARGINS

In this section we prove that

$$\text{Trans } \mathbf{B} \neq \varnothing \Rightarrow (\forall \alpha)(\mathbf{B} \in \mathscr{P}(\alpha)), \tag{4.1}$$

i.e., $\mathbf{B} \in \mathscr{P}(\alpha)$ is a necessary condition for $\mathbf{B}$ to have a transversal. Theorem 1 says rather more than this.

THEOREM 1. *If* $\alpha \in \mathbf{On}$ *and* $\mathbf{B} = \langle B_i \mid i \in I \rangle$ *is a family of subsets of* X *and* $\Phi \in$ Trans $\mathbf{B}$, *then*

$$m_\alpha(X) \geqslant l(\Phi) \geqslant 0. \tag{4.2}$$

Remark. The theorem clearly implies (4.1). For if $\Phi \in$ Trans $\mathbf{B}$ and

$Y \subseteq X$, then Φ induces a transversal Φ' (say) on the family $\mathbf{B}' = \mathbf{B}(Y)$ of subsets of Y and hence, by the theorem and (2.8),

$$m_\alpha(Y) = m_\alpha(Y, \mathbf{B}(Y)) \geqslant l(\Phi', \mathbf{B}') \geqslant 0.$$

Proof of Theorem 1. If $m_\alpha(X) = \infty$, then (4.2) holds. Therefore we may assume that

$$m_\alpha(X) = k < \infty. \tag{4.3}$$

We prove the theorem by transfinite induction on α.

Case 1. If $\alpha = 0$, then (4.3) implies that X is finite and

$$k = m_0(X) = |X| - |\mathbf{B}| = l(\Phi) \geqslant 0.$$

Case 2. α is a limit ordinal. If $p > k$, then there is $\beta < \alpha$ such that $m_\beta(X) < p$. Therefore, by the induction hypothesis,

$$p > l(\Phi).$$

This holds for any $p > k$ and hence $k \geqslant l(\Phi)$.

Case 3. $\alpha = \beta + 1$. We first prove the weaker result

$$k \geqslant 0. \tag{4.4}$$

By (4.3) $\mathscr{A}_\beta(X) \neq \varnothing$. Consider any tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(X)$ with

$$m_\beta(T_n) = s_\beta(\mathbf{T}) < \infty.$$

Let $Q = \Phi(D(\mathbf{T}))$. The transversal Φ of $\mathbf{B}$ induces a transversal

$$\Phi_n = \Phi \restriction I(T_n)$$

of the subsystem $\mathbf{B}(T_n) = \mathbf{B} \restriction I(T_n)$. Since the elements of Q are not used as representatives of the members of $\mathbf{B}(T_n)$, it follows by the induction hypothesis that

$$|Q \cap T_n| \leqslant l(\Phi_n, \mathbf{B}(T_n)) \leqslant m_\beta(T_n, \mathbf{B}(T_n)) = m_\beta(T_n) = s_\beta(\mathbf{T}).$$

This holds for each $n < \omega$ and so $|Q| \leqslant s_\beta(\mathbf{T})$. But $|Q| = d(\mathbf{T})$. Therefore,

$$s_\beta(\mathbf{T}) - d(\mathbf{T}) \geqslant 0.$$

This inequality holds for every $\mathbf{T} \in \mathscr{A}_\beta(X)$ and hence $k = m_{\beta+1}(X) \geqslant 0$. This completes the proof of (4.4).

We now deduce (4.2). Consider any finite set $P \subseteq X - \Phi(I)$. Since $\Phi \in \text{Trans}(\mathbf{B} \backslash P)$, it follows from (4.4) (applied to the system $\mathbf{B} \backslash P$) and Lemma 3.4 that

$$m_\alpha(X) - |P| = m_\alpha(X - P, \mathbf{B} \backslash P) \geqslant 0.$$

Therefore, $m_\alpha(X) \geqslant |P|$ whenever P is finite and $|P| \leqslant l(\Phi)$, and (4.2) follows.

5. Margins Bounded Below

In this section we prove some further results about the margin functions on the assumption that they are bounded below. In [8] Nash-Williams stated the conjecture that m_α is submodular, i.e.,

$$m_\alpha(Y \cup Z) + m_\alpha(Y \cap Z) \leqslant m_\alpha(Y) + m_\alpha(Z) \tag{5.1}$$

holds for $\alpha \in \mathbf{On}$ and $Y, Z \subseteq X$. Lemma 5.1 shows that the stronger inequality (5.3) holds under the hypothesis (5.2). We do not know if (5.2) can be replaced by the weaker condition $m_\alpha(W) > -\infty$ $(\forall W \subseteq Y \cup Z)$, but the following example shows that some restriction must be placed upon the system $\mathbf{B}$ in order to prove (5.1).

Example. Let $U = \{u_i \mid i < \omega\}$, $V = \{v_i \mid i < \omega\}$, $W = \{w_i \mid i < \omega\}$ be mutually disjoint denumerable sets, and let $\mathbf{B}$ be the system

$$\langle \{u_i\} \mid i < \omega \rangle + \langle \{w_i\} \mid i < \omega \rangle + \langle \{u_i, v_i\} \mid i < \omega \rangle + \langle \{v_i, w_i\} \mid i < \omega \rangle.$$

Put $Y = U \cup V$, $Z = V \cup W$. It is easy to see that $m_1(Y) = m_1(Z) = 0$, and $m_1(Y \cap Z) = m_1(V) = \infty$ since $\mathscr{A}_0(V) = \varnothing$. For $n < \omega$ let $\mathbf{T}_n$ be the tower $\langle T_{nj} \mid j < \omega \rangle$, where $T_{nj} = \{u_i \mid i < n + j\} \cup \{v_i \mid i < n + 2j\} \cup \{w_i \mid i < n + j\}$.

Then

$$m_0(T_{nj}) = (3n + 4j) - 4(n + j) = -n \qquad (\forall j < \omega).$$

Therefore, $\mathbf{T}_n \in \mathscr{A}_0(Y \cup Z)$ and $m_1(Y \cup Z) \leqslant s_0(\mathbf{T}_n) = -n (\forall n < \omega)$. Therefore $m_1(Y \cup Z) = -\infty$ and the left side of (5.1) is meaningless.

Lemma 5.1. *Let $\alpha \in \mathbf{On}$, $h < \omega$, $Y \cup Z \subseteq X$ and suppose that*

$$m_\alpha(W) \geqslant -h \qquad (\forall W \subseteq Y \cup Z). \tag{5.2}$$

Then

$$m_\alpha(Y \cup Z) + m_\alpha(Y \cap Z) + |\, I(Y \cup Z) - I(Y) - I(Z)| \leqslant m_\alpha(Y) + m_\alpha(Z). \tag{5.3}$$

Proof. For $Y, Z \subseteq X$, let

$$M(Y, Z, \mathbf{B}) = I(Y \cup Z) - I(Y) - I(Z).$$

We prove the lemma by transfinite induction on α.

Case 1. $\alpha = 0$. Since $I(Y \cup Z)$ is the disjoint union of $I(Y)$, $I(Z) - I(Y \cap Z)$ and $M(Y, Z)$, we have

$$|\, \mathbf{B}(Y \cup Z)| + |\, \mathbf{B}(Y \cap Z)| - |\, M(Y, Z)| = |\, \mathbf{B}(Y)| + |\, \mathbf{B}(Z)|$$

and (5.3) follows from the definition of m_0.

Case 2. α a limit ordinal. By the inductive hypothesis,

$$m_\beta(Y \cup Z) + m_\beta(Y \cap Z) - |\, M(Y, Z)| \leqslant m_\beta(Y) + m_\beta(Z)$$

holds for all $\beta < \alpha$ and (5.3) follows by continuity (2.4).

Case 3. $\alpha = \beta + 1$. If either $m_\alpha(Y) = \infty$ or $m_\alpha(Z) = \infty$, then (5.3) holds trivially. Therefore we may assume that

$$-2h \leqslant m_\alpha(Y) + m_\alpha(Z) < \infty.$$

By (2.2) there are towers $\mathbf{S} = \langle S_n \rangle \in \mathscr{A}_\beta(Y)$, $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\beta(Z)$ such that

$$m_\alpha(Y) = s_\beta(\mathbf{S}) - d(\mathbf{S}), \qquad m_\alpha(Z) = s_\beta(\mathbf{T}) - d(\mathbf{T}). \tag{5.4}$$

Put $U_n = S_n \cup T_n$, $V_n = S_n \cap T_n$, $\mathbf{U} = \langle U_n \rangle$.

Using the functions F, D defined in Section 2, we now define the following subsets of I:

$$\begin{aligned}
J_0 &= D(\mathbf{S}) \cap D(\mathbf{T}),\\
J_1 &= D(\mathbf{S}) \cap F(\mathbf{T}),\\
J_2 &= F(\mathbf{S}) \cap D(\mathbf{T}),\\
J_3 &= D(\mathbf{S}) - I(Z),\\
J_4 &= D(\mathbf{T}) - I(Y),\\
M_0 &= M(Y, Z) \cap D(\mathbf{U})\\
M_1 &= M(Y, Z) \cap F(\mathbf{U}).
\end{aligned}$$

It is easily seen that these seven sets are pairwise disjoint,

$$M_0 \cup M_1 = M(Y, Z)$$

and

$$D(\mathbf{S}) \cup D(\mathbf{T}) = \bigcup_{n \leqslant 4} J_n = J.$$

Since the margins (5.4) are finite so are $D(\mathbf{S})$, $D(\mathbf{T})$ and J. Put

$$\mathbf{B}' = \mathbf{B} \upharpoonright I - J.$$

By Lemma 3.3 and (2.3),

$$-h \leqslant m_\alpha(W, \mathbf{B}) \leqslant m_\alpha(W, \mathbf{B}') \leqslant m_\beta(W, \mathbf{B}') \qquad (\forall W \subseteq Y \cup Z).$$

Therefore, by the induction hypothesis applied to $(S_n, T_n, \mathbf{B}')$ and Lemma 3.5, we have

$$\begin{aligned} -2h &\leqslant m_\beta(U_n, \mathbf{B}') + m_\beta(V_n, \mathbf{B}') + | M(S_n, T_n, \mathbf{B}')| \\ &\leqslant m_\beta(S_n, \mathbf{B}') + m_\beta(T_n, \mathbf{B}') \\ &= m_\beta(S_n) + | J(S_n)| + m_\beta(T_n) + | J(T_n)| \\ &\leqslant s_\beta(\mathbf{S}) + s_\beta(\mathbf{T}) + | J_2 | + | J_1 | = k, \qquad \text{say}, \end{aligned} \tag{5.5}$$

since $J(S_n) \subseteq J_2$, $J(T_n) \subseteq J_1$. Now k does not depend upon n and so there are m, u, v and an infinite set $R \subseteq \omega$ such that

$$\begin{gathered} -h \leqslant m_\beta(U_n, \mathbf{B}') = u < \infty, -h \leqslant m_\beta(V_n, \mathbf{B}') = v < \infty, \\ 0 \leqslant | M(S_n, T_n, \mathbf{B}')| = m < \infty \qquad (\forall n \in R). \end{gathered} \tag{5.6}$$

By the definition of M_1 we see that

$$M_1 \subseteq \bigcup_{n \in R} M(S_n, T_n)$$

and

$$M_1 \cap M(S_n, T_n) \subseteq M_1 \cap M(S_{n+1}, T_{n+1}) \qquad (\forall n < \omega).$$

Also,

$$M(S_n, T_n, \mathbf{B}') = M(S_n, T_n, \mathbf{B})$$

since $J \cap M(Y, Z) = \varnothing$. It follows that

$$| M_1 | \leqslant \max_{n \in R} | M(S_n, T_n, \mathbf{B}')| = m.$$

Therefore, by (5.6), (5.5) and (5.4), we have

$$\begin{aligned} u + v + |M_1| &\leqslant s_\beta(\mathbf{S}) + s_\beta(\mathbf{T}) + |J_1| + |J_2| \\ &= m_\alpha(Y) + m_\alpha(Z) + d(\mathbf{S}) + d(\mathbf{T}) + |J_1| + |J_2| \\ &= m_\alpha(Y) + m_\alpha(Z) + (|J_0| + |J_1| + |J_3|) \\ &\quad + (|J_0| + |J_2| + |J_4|) + |J_1| + |J_2| \\ &= m_\alpha(Y) + m_\alpha(Z) + |J| + |J_0| + |J_1| + |J_2|. \end{aligned} \tag{5.7}$$

By (5.6), $\mathbf{U}^* = \langle U_n \mid n \in R\rangle \in \mathscr{A}_\beta(Y \cup Z, \mathbf{B}')$. Therefore, since

$$M_0 \subseteq D(\mathbf{U}) = D(\mathbf{U}^*),$$

we have

$$m_\alpha(Y \cup Z, \mathbf{B}') \leqslant s_\beta(\mathbf{U}^*, \mathbf{B}') - d(\mathbf{U}^*, \mathbf{B}') \leqslant u - |M_0|.$$

Also, $\mathbf{V}^* = \langle V_n \mid n \in R\rangle \in \mathscr{A}_\beta(Y \cap Z, \mathbf{B}')$ and so

$$m_\alpha(Y \cap Z, \mathbf{B}') \leqslant s_\beta(\mathbf{V}^*, \mathbf{B}') = v.$$

It follows from these inequalities and Lemma 3.5 and (5.7) that

$$\begin{aligned} &m_\alpha(Y \cup Z) + m_\alpha(Y \cap Z) + |M(Y, Z)| \\ &\quad \leqslant m_\alpha(Y \cup Z, \mathbf{B}') + m_\alpha(Y \cap Z, \mathbf{B}') - |J(Y \cup Z)| - |J(Y \cap Z)| \\ &\qquad + |M_0| + |M_1| \\ &\quad \leqslant u - |M_0| + v - |J| - |J_0 \cup J_1 \cup J_2| + |M_0| + |M_1| \\ &\quad \leqslant m_\alpha(Y) + m_\alpha(Z). \end{aligned}$$

This completes the proof of the lemma.

LEMMA 5.2. *Let $\alpha \in \mathbf{On}$, $A \subseteq X$, $\mathbf{B} \in \mathscr{P}(\alpha)$ and*

$$m_\alpha(Y) \geqslant k \qquad (\forall Y)(A \subseteq Y \subseteq X). \tag{5.8}$$

Suppose that $A \subseteq Y_i \subseteq X$ and $m_\alpha(Y_i) = k$ ($\forall i < \omega$). Then

$$m_\alpha\left(\bigcup_{i \leqslant n} Y_i\right) = k \qquad (n < \omega), \tag{5.9}$$

and

$$m_{\alpha+1}\left(\bigcup_{i<\omega} Y_i\right) \leqslant k. \tag{5.10}$$

Proof. Since $\mathbf{B} \in \mathscr{P}(\alpha)$, $k \geqslant 0$. By (5.8) and Lemma 5.1,

$$2k \leqslant m_\alpha(Y_0 \cup Y_1) + m_\alpha(Y_0 \cap Y_1) \leqslant m_\alpha(Y_0) + m_\alpha(Y_1) = 2k.$$

Therefore, by (5.8), $m_\alpha(Y_0 \cup Y_1) = m_\alpha(Y_0 \cap Y_1) = k$. Now (5.9) follows by induction on n.

Put $S_n = \bigcup_{i \leqslant n} Y_i$, $\mathbf{S} = \langle S_n \mid n < \omega \rangle$. Then $\mathbf{S}$ is a tower under

$$Y^* = \bigcup_{i<\omega} Y_i$$

with $m_\alpha(S_n) = k (\forall n < \omega)$, and so $m_{\alpha+1}(Y^*) \leqslant k$.

LEMMA 5.3. *Let $\mathbf{B} \in \mathscr{P}(\Omega)$, $B \in \mathbf{B}$, $|B| \leqslant \aleph_0$. Then there is an element $b \in B$ such that*

$$m_\Omega(Y, \mathbf{C}_b) \geqslant 0 \qquad (\forall Y)(Y \subset X - \{b\}), \tag{5.11}$$

where $\mathbf{C}_b = (\mathbf{B} - \langle B \rangle) \backslash \{b\}$. In other words, $\mathbf{C}_b \in \mathscr{P}(\Omega)$.

Proof. Suppose the lemma is false. Then for each $b \in \mathbf{B}$ there is a set $Y_b \subseteq X - \{b\}$ such that

$$m_\Omega(Y_b, \mathbf{C}_b) < 0. \tag{5.12}$$

Put $\mathbf{B}' = \mathbf{B} - \langle B \rangle$, $Z_b = Y_b \cup \{b\}$. By (5.12) and Lemma 3.4,

$$m_\Omega(Z_b, \mathbf{B}') = m_\Omega(Y_b, \mathbf{C}_b) + 1 < 1.$$

By Lemma 3.3, $\mathbf{B}' \in \mathscr{P}(\Omega)$. Therefore,

$$m_\alpha(Z_b, \mathbf{B}') = 0 \qquad (\forall b \in B). \tag{5.13}$$

Case 1. B is covered by finitely many of the sets Z_b. Say

$$B \subseteq K = Z_{b_1} \cup \cdots \cup Z_{b_n},$$

where $n < \omega$ and $\{b_1, \ldots, b_n\} \subseteq B$. By (5.13) and Lemma 5.2

$$m_\Omega(K, \mathbf{B}') = 0.$$

Then by Lemma 3.5,

$$m_\Omega(K, \mathbf{B}) = -1$$

since $B \subseteq K$. This contradicts the hypothesis that $\mathbf{B} \in \mathscr{P}(\Omega)$. So Case 1 is impossible.

Case 2. B is not covered by any finite union of the sets Z_b. Then B is denumerable, say $B = \{b_i \mid i < \omega\}$. Since $b \in Z_b$, we have

$$B \subseteq K = \bigcup_{i<\omega} Z_{b_i}.$$

By (5.13) and Lemma 5.2,

$$m_{\Omega+1}(K, \mathbf{B}') \leqslant 0.$$

Now Lemmas 3.1 and 3.5 yield the contradiction

$$m_{\Omega}(K, \mathbf{B}) = m_{\Omega+1}(K, \mathbf{B}) = m_{\Omega+1}(K, \mathbf{B}') - 1 < 0.$$

Thus Case 2 is also impossible. This contradiction proves the lemma.

6. Sufficiency of nonnegative Margins

In this section we prove that a countable system of sets $\mathbf{B}$ has a transversal if and only if $\mathbf{B} \in \mathscr{P}(\Omega)$. Our main result, Theorem 2, says more than this.

THEOREM 2. *Let* $\mathbf{B}$ *be a countable system of subsets of* X. *Then*

$$\mathrm{Trans}(\mathbf{B}) \neq \varnothing \Leftrightarrow \mathbf{B} \in \mathscr{P}(\Omega). \tag{6.1}$$

Also, if $\mathbf{B} \in \mathscr{P}(\Omega)$ *and* $m_\Omega(X) = k$, *then*

$$(\forall \Phi)(\Phi \in \mathrm{Trans}(\mathbf{B}) \Rightarrow l(\Phi) \leqslant k) \tag{6.2}$$

$$(\exists \Phi)(\Phi \in \mathrm{Trans}(\mathbf{B}) \;\&\; l(\Phi) = k). \tag{6.3}$$

Proof of (6.1). If $\mathrm{Trans}(\mathbf{B}) \neq \varnothing$, then $\mathrm{Trans}(\mathbf{B}(Y)) \neq \varnothing$ for all $Y \subseteq X$ and therefore $m_\Omega(Y) \geqslant 0$ by Theorem 1. To complete the proof of (6.1) we must prove the implication in the opposite direction, i.e.,

$$\mathbf{B} \in \mathscr{P}(\Omega) \Rightarrow \mathrm{Trans}(\mathbf{B}) \neq \varnothing. \tag{6.4}$$

Suppose $\mathbf{B} = \langle B_i \mid i \in I \rangle \in \mathscr{P}(\Omega)$. Let $I_1 = \{i \in I \mid |B_i| \leqslant \aleph_0\}$, $\mathbf{B}_1 = \mathbf{B} \restriction I_1$. By Lemma 3.3 we have

$$\mathbf{B}_1 \in \mathscr{P}(\Omega). \tag{6.5}$$

It will be enough to prove that $\mathrm{Trans}(\mathbf{B}_1) \neq \varnothing$. For if $\Phi \in \mathrm{Trans}(\mathbf{B}_1)$ and $I - I_1 = \{i_n \mid n < s\}$, where $s \leqslant \omega$, then we can successively choose representatives $x_n \in B_{i_n} - (\Phi(I_1) \cup \{x_j \mid j < n\})$ for all $n < s$ for the uncountable members of $\mathbf{B}$.

Since $|I_1| \leqslant \aleph_0$, we may assume that $I_1 = \{n \mid n < t\}$ for some $t \leqslant \omega$. We will construct a transversal of $\mathbf{B}_1 = \langle B_n \mid n < t\rangle$ by successively choosing distinct representatives $b_n \in B_n$ $(n < t)$ in the following way. Let $n < t$ and suppose we have already chosen $b_i \in B_i$ for $i < n$ so that

$$b_i \neq b_j \qquad \text{if} \quad i < j < n \tag{6.6}$$

and

$$\mathbf{C}_n = (\mathbf{B}_1 - \langle B_i \mid i < n\rangle)\backslash\{b_i \mid i < n\} \in \mathscr{P}(\Omega). \tag{6.7}$$

Note that $\mathbf{C}_0 = \mathbf{B}_1 \in \mathscr{P}(\Omega)$. Put $B_i^n = B_i - \{b_j \mid j < n\}$. Then

$$\mathbf{C}_n = \langle B_i^n \mid n \leqslant i < t\rangle.$$

By Lemma 5.3 there is an element $b_n \in B_i^n$ such that

$$(\mathbf{C}_n - \langle B_n^n\rangle)\backslash\{b_n\} \in \mathscr{P}(\Omega).$$

Thus (6.6) and (6.7) hold with n replaced by $n + 1$ since

$$(\mathbf{C}_n - \langle B_n^n\rangle)\backslash\{b_n\} = (\mathbf{B}_1 - \langle B_i \mid i \leqslant n\rangle)\backslash\{b_i \mid i \leqslant n\} = \mathbf{C}_{n+1}.$$

The sequence $\langle b_n \mid n < t\rangle$ defined inductively in this way is a system of distinct representatives for $\mathbf{B}_1$. This shows that $\mathrm{Trans}(\mathbf{B}_1) \neq \varnothing$ and hence $\mathrm{Trans}(\mathbf{B}) \neq \varnothing$. This proves (6.4) and hence (6.1).

Proof of (6.2). If $k = \infty$ (6.2) holds trivially and if $k < \infty$ then (6.2) follows from Theorem 1.

Proof of (6.3). Suppose $\mathbf{B} \in \mathscr{P}(\Omega)$ and $m_\Omega(X) = k$. Put $X_n = X$ for $n < k$ and consider the system

$$\mathbf{B}^* = \mathbf{B} + \langle X_n \mid 0 \leqslant n < k\rangle$$

obtained by adjoining k copies of X to $\mathbf{B}$. We will show that

$$\mathbf{B}^* \in \mathscr{P}(\Omega). \tag{6.8}$$

Since $\mathbf{B}^*(Y) = \mathbf{B}(Y)$ if Y is a proper subset of X, we have

$$m_\Omega(Y, \mathbf{B}^*) = m_\Omega(Y, \mathbf{B}) \geqslant 0 \qquad (\forall Y)(Y \subset X).$$

It remains to show that

$$m_\Omega(X, \mathbf{B}^*) \geqslant 0. \tag{6.9}$$

Case 1. $k < \infty$. Then by Lemma 3.5,

$$m_\Omega(X, \mathbf{B}^*) = m_\Omega(X, \mathbf{B}) - k = 0.$$

Case 2. $k = \infty$. Then $m_\Omega(X) = k$ implies that X is an infinite set. Hence $m_0(X, \mathbf{B}^*) = \infty$. Suppose that (6.9) is false. Then there is a least ordinal α such that $0 < \alpha < \Omega$ and

$$m_\alpha(X, \mathbf{B}^*) < \infty. \tag{6.10}$$

From the definition of m_α it follows that α is not a limit ordinal. Hence $\alpha = \beta + 1$ and $m_\beta(X, \mathbf{B}^*) = \infty$. By (6.10) there is a tower

$$\mathbf{T} = \langle T_j \mid j < \omega \rangle \in \mathscr{A}_\beta(X, \mathbf{B}^*)$$

such that $s(\mathbf{T}, \mathbf{B}^*) = m_\beta(T_j, \mathbf{B}^*) = s < \infty$ $(\forall j < \omega)$. Since

$$m_\beta(T_j, \mathbf{B}^*) < \infty,$$

T_n is a proper subset of X. Hence $m_\beta(T_j, \mathbf{B}) = m_\beta(T_j, \mathbf{B}^*) = s < \infty$. Thus $\mathbf{T} \in \mathscr{A}_\beta(X, \mathbf{B})$ and

$$m_\alpha(X, \mathbf{B}) \leqslant s - d(\mathbf{T}, \mathbf{B}) < \infty.$$

This is a contradiction since $m_\Omega(X) \leqslant m_\alpha(X)$ by (2.3). This proves (6.9) and hence (6.8) holds. By (6.1), $\mathbf{B}^*$ has a transversal and this induces a transversal Φ, say, of $\mathbf{B}$ with at least k elements left over, i.e. $l(\Phi) \geqslant k$. But $l(\Phi) \leqslant k$ by (6.2) and so $l(\Phi) = k$ as required.

7. Examples

In this section examples are given to show that Theorem 2 is a best possible result in two senses. Example 1 shows that the hypothesis $|\mathbf{B}| \leqslant \aleph_0$ cannot be dropped from Theorem 2 and Example 2 shows that the condition $\mathscr{P}(\Omega)$ cannot be replaced by $\mathscr{P}(\alpha)$ for any $\alpha < \Omega$.

As usual, an ordinal number α is identified with the set of all smaller ordinals, $\alpha = \{\beta \mid \beta < \alpha\}$.

Example 1. Let $\mathbf{B} = \langle \alpha \mid \omega \leqslant \alpha < \Omega \rangle$, an uncountable system of denumerable sets. We first show that $\mathbf{B} \in \mathscr{P}(\Omega)$. Let Y be any set of ordinals. If $|Y| > \aleph_0$, then $m_\Omega(Y) = \infty$ by Lemma 3.2. If $|Y| \leqslant \aleph_0$, $\mathbf{B}(Y)$ is a countable system of denumerable sets and trivially any such family has a transversal. Hence $m_\Omega(Y) \geqslant 0$ by Theorem 1. This proves that $\mathbf{B} \in \mathscr{P}(\Omega)$. However, by a well-known theorem on regressive functions [1], $\text{Trans}(\mathbf{B}) = \varnothing$ (for if $\Phi(\alpha) \in \alpha$ $(\omega \leqslant \alpha < \Omega)$, then there is $\Theta < \Omega$ such that $|\{\alpha \mid \omega \leqslant \alpha < \Omega \;\&\; \Phi(\alpha) = \Theta\}| = \aleph_1$).

EXAMPLE 2. Let $\alpha < \Omega$ and let

$$\mathbf{B}^\alpha = \langle \beta + 1 \mid \beta < \omega(1 + \alpha)\rangle, \qquad \mathbf{B} = \mathbf{B}^\alpha + \langle \omega(1 + \alpha)\rangle.$$

It is easy to see that $\mathbf{B}^\alpha$ has the unique transversal Ξ, where $\Xi(\beta + 1) = \beta (\forall \beta < \omega(1 + \alpha))$, and hence Trans $(\mathbf{B}) = \varnothing$. We will show that $\mathbf{B} \in \mathscr{P}(\alpha)$ (and $\mathbf{B} \notin \mathscr{P}(\alpha + 1)$). In order to prove this, we need a lemma which describes the margin functions for $\mathbf{B}^\alpha$.

For any set of ordinals Y, we shall denote by $\gamma(Y)$ the least ordinal $\gamma \notin Y$ and put $r(Y) = Y - \gamma(Y)$. Thus Y is the disjoint union of the sets $\gamma(Y) \cup r(Y)$. In particular, for any ordinal α, $\gamma(\alpha) = \alpha$ and $r(\alpha) = \varnothing$.

LEMMA 7.1. *Let $\lambda \leqslant \alpha < \Omega$, $Y \subseteq \Omega$. Then*

$$m_\lambda(Y, \mathbf{B}^\alpha) = \begin{cases} |r(Y)| & \text{if } \gamma(Y) < \omega(1 + \lambda), \\ \infty & \text{if } \gamma(Y) \geqslant \omega(1 + \lambda). \end{cases} \tag{7.1}$$

Proof. Our proof will be by induction on λ. We write $m_\lambda(Y)$ for $m_\lambda(Y, \mathbf{B}^\alpha)$.

Case 1. $\lambda = 0$.

(a) Suppose $\gamma(Y) < \omega$. If Y is infinite, then so is $r(Y)$ and

$$m_0(Y) = \infty = |r(Y)|.$$

If Y is finite, then $m_0(Y) = |Y| - |\mathbf{B}^\alpha(Y)| = |Y| - |\gamma(Y)| = |r(Y)|$.

(b) Suppose $\gamma(Y) \geqslant \omega$. Then Y is infinite and $m_0(Y) = \infty$.

Case 2. λ is a limit ordinal.

(a) If $\gamma(Y) < \omega(1 + \lambda)$, then there is a least ordinal $\lambda_0 < \lambda$ such that $\gamma(Y) < \omega(1 + \lambda_0)$. Then, by the induction hypothesis,

$$m_\mu(Y) = \infty (\mu < \lambda_0), \qquad m_\mu(Y) = |r(Y)| (\lambda_0 \leqslant \mu < \lambda)$$

and hence $m_\lambda(Y) = \inf_{\mu < \lambda} m_\mu(Y) = |r(Y)|$.

(b) If $\gamma(Y) \geqslant \omega(1 + \lambda)$, then $m_\mu(Y) = \infty$ for all $\mu < \lambda$ and hence $m_\lambda(Y) = \infty$.

Case 3. $\lambda = \mu + 1$.

Case 3a. $\gamma(Y) < \omega(1 + \lambda)$. We will first prove that

$$m_\lambda(Y) \leqslant |\, r(Y)| . \tag{7.2}$$

If $\gamma(Y) < \omega(1 + \mu)$, then (7.2) is obvious since, by (2.3) and the induction hypothesis,

$$m_\lambda(Y) \leqslant m_\mu(Y) = |\, r(Y)| .$$

Suppose now that $\omega(1 + \mu) \leqslant \gamma(Y) < \omega(1 + \lambda)$. Put

$$Z = \{\beta \mid \omega(1 + \mu) \leqslant \beta < \gamma(Y)\}.$$

Then $Y = \omega(1 + \mu) \cup Z \cup r(Y)$. Let $\langle \gamma_n \mid n < \omega \rangle$ be any sequence of ordinals such that $\gamma_0 < \gamma_1 < \cdots < \omega(1 + \mu) = \lim \gamma_n$, and put

$$T_n = \gamma_n \cup Z \cup r(Y).$$

Then $\mathbf{T} = \langle T_n \mid n < \omega \rangle$ is a tower under Y and, by the induction hypothesis,

$$m_\mu(T_n) = |\, r(T_n)| = |\, Z\,| + |\, r(Y)| .$$

If $|\, r(Y)| = \infty$, then (7.2) holds trivially and so we may assume that $|\, r(Y)| < \infty$. Then $m_\mu(T_n)$ is a finite constant which does not depend upon n. Since $\beta + 1 \in D(\mathbf{T})(\forall \beta \in Z)$, it follows that

$$m_\lambda(Y) \leqslant |\, Z\,| + |\, r(Y)| - |\, D(\mathbf{T})| \leqslant |\, r(Y)|.$$

This completes the proof of (7.2).

We now prove the reverse inequality

$$m_\lambda(Y) \geqslant |\, r(Y)| . \tag{7.3}$$

If $m_\lambda(Y) = \infty$, then (7.3) holds trivially. So we assume that $m_\lambda(Y) < \infty$ and hence that $\mathscr{A}_\mu(Y) \neq \varnothing$. Consider any tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\mu(Y)$. Let

$$m_\mu(T_n) = s_\mu(\mathbf{T}) = s < \infty. \tag{7.4}$$

By (7.4) and the induction hypothesis, it follows that $\gamma(T_n) < \omega(1 + \mu)$ and $|\, r(T_n)| = s$ $(n < \omega)$. Let $D = D(\mathbf{T})$. If $\beta \in D$, then $\beta + 1 \subseteq Y$ and $\beta + 1 \nsubseteq T_n$ and hence

$$\gamma(T_n) < \beta + 1 \leqslant \gamma(Y) \qquad (n < \omega).$$

Therefore $D \subseteq Y \backslash r(Y)$ and $D \cap T_n \subseteq r(T_n) \quad (n < \omega)$. If W is any finite subset of $D \cup r(Y)$, then $W \subseteq r(T_n)$ for some $n < \omega$ and hence, $|W| \leqslant s$. Therefore $D \cup r(Y)$ is a finite set and $D \cup r(Y) \subseteq r(T_{n_0})$ for some $n_0 < \omega$. Therefore,

$$s_\mu(\mathbf{T}) - d(\mathbf{T}) \geqslant |D \cup r(Y)| - |D| = |r(Y)|.$$

Since this holds for any tower $\mathbf{T} \in \mathscr{A}_\mu(Y)$, (7.3) follows and the proof of (7.1) is complete for this case.

Case 3b. $\gamma(Y) \geqslant \omega(1 + \lambda)$. Suppose there is a tower $\mathbf{T} = \langle T_n \rangle \in \mathscr{A}_\mu(Y)$. Let $m_\mu(T_n) = s_\mu(\mathbf{T}) = s < \infty$ Then, by the induction hypothesis, $\gamma(T_n) < \omega(1 + \mu)$ and $|r(T_n)| = s$. There is a finite set

$$W \subseteq \{\beta \mid \omega(1 + \mu) \leqslant \beta < \omega(1 + \lambda)\}$$

such that $|W| > s$. Since $\gamma(Y) \geqslant \omega(1 + \lambda)$, we have $W \subseteq Y$. Hence there is $n < \omega$ such that $W \subseteq T_n$ and since $\gamma(T_n) < \omega(1 + \mu)$, we have $W \subseteq r(T_n)$. This is impossible. Therefore $\mathscr{A}_\mu(Y) = \varnothing$ and $m_\mu(Y) = \infty$. This completes the proof of Lemma 7.1.

Now consider the system $\mathbf{B} = \mathbf{B}^\alpha + \langle \omega(1 + \alpha) \rangle$. Let $Y \subseteq \Omega$. We want to show that

$$m_\alpha(Y, \mathbf{B}) \geqslant 0. \tag{7.5}$$

If $\omega(1 + \alpha) \subseteq Y$, then $\mathbf{B}(Y) = \mathbf{B}^\alpha(Y) + \langle \omega(1 + \alpha) \rangle$ and so, by Lemma 3.5,

$$m_\alpha(Y, \mathbf{B}) = m_\alpha(Y, \mathbf{B}^\alpha) - 1.$$

But, by Lemma 7.1, $m_\alpha(Y, \mathbf{B}^\alpha) = \infty$ and so (7.5) holds in this case. Now suppose that $\omega(1 + \alpha) \nsubseteq Y$. Then $\mathbf{B}(Y) = \mathbf{B}^\alpha(Y)$ and so $m_\alpha(Y, \mathbf{B}) = m_\alpha(Y, \mathbf{B}^\alpha) \geqslant 0$. Thus (7.5) holds and $\mathbf{B} \in \mathscr{P}(\alpha)$.

As we have already remarked, the system $\mathbf{B}$ does not have a transversal. We now prove the stronger assertion that

$$\mathbf{B} \notin \mathscr{P}(\alpha + 1).$$

By the lemma, $m_{\alpha+1}(\omega(1 + \alpha), \mathbf{B}^{\alpha+1}) = 0$. Therefore, since

$$\mathbf{B}^{\alpha+1}(\omega(1 + \alpha)) + \langle \omega(1 + \alpha) \rangle = \mathbf{B}(\omega(1 + \alpha)),$$

it follows by Lemma 3.5, that

$$m_{\alpha+1}(\omega(1 + \alpha), \mathbf{B}) = -1.$$

Hence $\mathbf{B} \notin \mathscr{P}(\alpha + 1)$.

8. Systems with Countably Many Denumerable Sets

For brevity, we write $\mathbf{B} \in \mathscr{D}_\alpha$ if $\mathbf{B} = \langle B_i \mid i \in I \rangle$ is a system of sets such that (i)$| B_i | \leqslant \aleph_\alpha (\forall i \in I)$ and (ii)$|\{i \in I \mid | B_i | \geqslant \aleph_0\}| \leqslant \aleph_\alpha$. We shall prove (Theorem 4) that a system $\mathbf{B} \in \mathscr{D}_0$ has a transversal if and only if $\mathbf{B} \in \mathscr{P}(\Omega)$.

Also we write $\mathbf{B} \in \mathscr{F}$ if $\mathbf{B}$ has only finitely many infinite members and $\mathbf{B} \in \mathscr{F}_0$ if $\mathbf{B} \in \mathscr{F}$ and every $B_i \in \mathbf{B}$ is countable (that is, $\mathscr{F}_0 = \mathscr{F} \cap \mathscr{D}_0$). Necessary and sufficient conditions have been given [2, 3] for a system $\mathbf{B} \in \mathscr{F}$ to have a transversal. Nash-Williams conjectured that if $\mathbf{B} \in \mathscr{F}$ and $| \mathbf{B} | \leqslant \aleph_0$, then

$$\text{Trans}\,(\mathbf{B}) \neq \varnothing \Leftrightarrow \mathbf{B} \in \mathscr{P}(1). \tag{8.1}$$

We will prove (Theorem 5) the stronger result that (8.1) holds whenever $\mathbf{B} \in \mathscr{F}_0$. (It is easy to see that this implies Nash-Williams' conjecture. For, if $\mathbf{B} \in \mathscr{F}$ and $| \mathbf{B} | \leqslant \aleph_0$, then $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where $\mathbf{B}_0 \in \mathscr{F}_0$ and $\mathbf{B}_1$ is a finite family of uncountable sets. Clearly $\mathbf{B}$ has a transversal if and only if $\mathbf{B}_0$ has.) We remark that (8.1) is not true for arbitrary $\mathbf{B} \in \mathscr{F}$. To see this consider the system

$$\mathbf{B} = \langle \{\alpha\} \mid \alpha < \Omega \rangle + \langle \Omega \rangle \in \mathscr{F}$$

consisting of the singletons $\{\alpha\}$ $(\alpha < \Omega)$ and just one uncountable set Ω. Clearly, Trans $(\mathbf{B}) = \varnothing$. Let Y be any set. If $| Y | > \aleph_0$, then $m_1(Y) = \infty$ by Lemma 3.2. If $| Y | \leqslant \aleph_0$, then $\mathbf{B}(Y)$ has a transversal and hence $m_1(Y) \geqslant 0$ by Theorem 1. Hence $\mathbf{B} \in \mathscr{P}(1)$ and (8.1) is false.

In order to prove our results we quote the following theorem from [7].

Theorem 3. *If* $\alpha \in \mathbf{On}$ *and* $\mathbf{B} \in \mathscr{D}_\alpha$ *then* Trans $(\mathbf{B}) \neq \varnothing$ *if and only if*

$$\text{Trans}\,(\mathbf{B}') \neq \varnothing \qquad (\forall \mathbf{B}')(\mathbf{B}' \subseteq \mathbf{B}\ \&\ | \mathbf{B}' | \leqslant \aleph_\alpha). \tag{8.2}$$

We now deduce the following.

Theorem 4. *If* $\mathbf{B}$ *is a system of sets and* $\mathbf{B} \in \mathscr{D}_0$, *then* Trans $\mathbf{B} \neq \varnothing$ *if and only if*

$$m_\Omega(Y) \geqslant 0, \qquad (\forall Y)(Y \subseteq X\ \&\ | Y | \leqslant \aleph_0) \tag{8.3}$$

or equivalently

$$\mathbf{B} \in \mathscr{P}(\Omega). \tag{8.4}$$

Proof. If $\mathbf{B}$ has a transversal, then (8.3) holds by Theorem 1. Conversely, (8.3) implies that (8.2) holds with $\alpha = 0$; so B has a transversal by Theorem 3. Finally, (8.3) is equivalent to (8.4) by Lemma 3.2.

LEMMA 8.1. *Let* $\mathbf{B} \in \mathscr{F}$, $\mathbf{B} \in \mathscr{P}(1)$. *Then*

$$m_\Omega(Y) = m_1(Y)$$

for any set $Y \subseteq X$.

Proof. By (2.6) it will be enough to prove that $m_2(Y) = m_1(Y)$ for all Y. Suppose this is false. Then there is a set Y such that $m_2(Y) < m_1(Y)$. Hence there is a tower $\mathbf{T} = \langle T_n \mid n < \omega \rangle \in \mathscr{A}_1(Y)$ such that

$$m_1(T_n) = s_1(\mathbf{T}) = s < \infty \qquad (\forall n < \omega) \tag{8.5}$$

and

$$s - d(\mathbf{T}) < m_1(Y). \tag{8.6}$$

Since $\mathbf{B} \in \mathscr{F}$ we can assume that the sets T_n in the tower each contain the same infinite members of $\mathbf{B}$, say k of them.

Since $\mathbf{B} \in \mathscr{P}(1)$, we have $s \geqslant 0$. Hence for $n < \omega$ there is a tower $\mathbf{T}_n = \langle T_{n,i} \mid i < \omega \rangle \in \mathscr{A}_0(T_n)$ such that

$$m_0(T_{n,i}) = s_n < \infty \qquad (i < \omega), \tag{8.7}$$

$$s = s_n - d(\mathbf{T}_n). \tag{8.8}$$

By (8.7), the sets $T_{n,i}$ are all finite and hence $d(\mathbf{T}_n) = k (n < \omega)$. Therefore, by (8.8), $s_n = s_0\ (n < \omega)$. Since $Y = \bigcup_n \bigcup_i T_{n,i}$ is denumerable, we can successively choose indices $i_n (n < \omega)$ so that

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots \subseteq Y = \bigcup_{n<\omega} S_n\,,$$

where $S_n = T_{n,i_n}$. Then $\mathbf{S} = \langle S_n \rangle \in \mathscr{A}_0(Y)$ and so

$$m_1(Y) \leqslant s_0 - d(\mathbf{S}).$$

But $d(\mathbf{S}) = k + d(\mathbf{T})$. Therefore,

$$m_1(Y) \leqslant s - d(\mathbf{T}) < m_1(Y).$$

This contradiction proves the lemma.

THEOREM 5. *Let* $\mathbf{B} \in \mathscr{F}_0$. *Then*

$$\text{Trans}\,(\mathbf{B}) \neq \varnothing \Leftrightarrow \mathbf{B} \in \mathscr{P}(1).$$

Proof. By Theorem 1, Trans(**B**) $\neq \varnothing \Rightarrow \mathbf{B} \in \mathscr{P}(1)$. Conversely, if $\mathbf{B} \in \mathscr{P}(1)$, then $\mathbf{B} \in \mathscr{P}(\Omega)$ by Lemma 8.1. Then **B** has a transversal by Theorem 4, since $\mathscr{F}_0 \subseteq \mathscr{D}_0$.

References

1. Alexandroff and Urjsohn, Mémoire sur les espaces topologiques compacts, *Verhandl. Nederl. Akad. Wetensch. Sect. I*, **14**, Nos. 1, 5, (1929).
2. R. A. Brualdi and E. B. Scrimger, Exchange Systems, Matchings and Transversals, *J. Combinatorial Theory*, **5** (1968), 244–257.
3. J. Folkman, Transversals of infinite families with finitely many infinite members, *J. Combinatorial Theory*, **9** (1970), 200–220.
4. Marshall Hall, Jr., Distinct representatives of subsets, *Bull. Amer. Math. Soc.* **54** (1948), 922–926.
5. P. Hall, On Representatives of Subsets, *J. London Math. Soc.* **10** (1935), 26–30.
6. D. König, Graphok es matrixok, *Mat. Fiz. Lapok* **38** (1931), 116–119 [Hungarian with German Summary].
7. E. C. Milner and S. Shelah, Some theorems on transversals, to appear in the Proceedings of the colloquium on Infinite and Finite Sets at Keszthély, 1973. Colloquia Mathematica Societatis János Bolyai.
8. C. St. J. A. Nash-Williams, Which infinite set-systems have transversals?—A possible approach, *in* "Proceedings of Conference on Combinatorics at Oxford (1972)," pp. 237–253, Institute of Mathematics and Applications, 1972.
9. D. R. Woodall, Two results on infinite transversals, pp. 341–350, Institute of Mathematical Applications, 1972.